

TA Session 2: Consumer Static Optimization

ECON 325, Spring 2024

Haoran Wang

UMD

February 2, 2024

- Review of consumer's optimization problem with log utility
- Policy exercise: labor income tax
 - lump-sum tax vs. proportional tax.
 - interested in how these tax instruments impact household decisions;
 - welfare analysis.

Review: Consumer Optimization

- Preferences

$$U(c, l) = \log c + \gamma \log l$$

- $\gamma > 0$: relative weight of consumption (“donuts”) vs leisure (“Youtube”);

- Budget constraint:

$$c = w(h - l)$$

where

- h is the max. hours available (a constant);
- $n \equiv h - l$ is the labor hours;
- w is the real wage rate.

Read: “total expenditure = total income”.

- Solution: Lagrange multiplier, end up with optimal consumption $c^*(w)$, optimal leisure $l^*(w)$ and optimal labor hour $n^*(w)$.

Review: Consumer Optimization

Some extra exercises:

- plot the indifference curve and budget constraint on a (l, c) plane;
- solve the problem for consumers with other utility functions, e.g.

$$u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} + \gamma \log l$$

$$c(c, l) = c + \log l$$

(You can make up these questions yourself.)

- What's the unit of budget constraint? Is it in terms of dollars (nominal) or donuts (real)?

Introducing Tax Instruments

- Government charges the consumer two types of taxes:
 - ① Lump-sum tax T : the government charges the consumer T donuts, regardless of his income;
 - ② Proportional income tax τ : for each donut the consumer earns, the government charges the consumer a fixed proportion τ of it.
- How can we express them in mathematics?
 - ① Lump-sum tax: now the new budget constraint becomes

$$c = w(h - l) - T$$

- ② Proportional income tax:

$$c = w(h - l) \cdot (1 - \tau)$$

- What's government tax revenue in both cases?
Lump-sum: T ;
Proportional: $\tau w(h - l)$.

Introducing Tax Instruments

- The formulation in the previous page assumes that the government can take only one tax instrument at a time;
- If the government charges both taxes simultaneously, what is the consumer's budget constraint?

$$c = w(h - l) \cdot (1 - \tau) - T$$

- Now I illustrate how to solve the consumer's optimization problem with this modified budget constraint.
- Do expect questions like this to appear in the exam!
- Our solution will be an optimal consumption function $c^*(w, \tau, T)$, an optimal leisure function $l^*(w, \tau, T)$, and a labor supply curve $n^*(w, \tau, T)$.

Solving Consumer's Problem with Two Taxes

The problem is

$$\max_{c, l} \log c + \gamma \log l$$

subject to

$$c = w(h - l) \cdot (1 - \tau) - T$$

Procedures:

- 1 Set up the Lagrangian:

$$\mathcal{L} = \log c + \gamma \log l - \lambda(c - w(h - l)(1 - \tau) + T)$$

- 2 List out first order conditions:

on c : $1/c - \lambda = 0$

on l : $\gamma/l - \lambda w(1 - \tau) = 0$

on λ : $c - w(h - l)(1 - \tau) + T = 0$

Solving Consumer's Problem with Two Taxes

- 3 Solve the system of three equations

$$1/c - \lambda = 0$$

$$\gamma/l - \lambda w(1 - \tau) = 0$$

$$c - w(h - l)(1 - \tau) + T = 0$$

The first equation allows us to replace λ with $1/c$ in the second equation:

$$\frac{\gamma}{l} = \frac{1}{c} w(1 - \tau) \implies \frac{\gamma c}{l} = w(1 - \tau)$$

Read: “MRS = relative price”.

Solving Consumer's Problem with Two Taxes

- 3 Solve the system of three equations

$$1/c - \lambda = 0$$

$$\gamma/l - \lambda w(1 - \tau) = 0$$

$$c - w(h - l)(1 - \tau) + T = 0$$

(cont.) with

$$\frac{\gamma c}{l} = w(1 - \tau) \implies l = \frac{\gamma c}{w(1 - \tau)}$$

we can use it to replace l in the third equation:

$$c - w \left(h - \frac{\gamma c}{w(1 - \tau)} \right) (1 - \tau) + T = 0$$

This is an equation with only one unknown: c .

Solving Consumer's Problem with Two Taxes

We end up with the optimal consumption function

$$c^*(w, \tau, T) = \frac{wh(1 - \tau) - T}{1 + \gamma}$$

optimal leisure function

$$l^*(w, \tau, T) = \frac{\gamma}{1 + \gamma}h - \frac{\gamma T}{(1 + \gamma)w(1 - \tau)}$$

and labor supply curve

$$n^*(w, \tau, T) = \frac{1}{1 + \gamma}h + \frac{\gamma T}{(1 + \gamma)w(1 - \tau)}$$

How Taxes Distort Consumer Behavior?

- No-tax benchmark: $T = 0, \tau = 0$,

$$\begin{aligned}c^*(w, 0, 0) &= \frac{wh}{1 + \gamma} \\l^*(w, 0, 0) &= \frac{\gamma}{1 + \gamma} h \\n^*(w, 0, 0) &= \frac{1}{1 + \gamma} h\end{aligned}$$

- Parameter γ pins down the fraction of work vs. leisure:
 - high γ , leisure valued more, hence work less and watch more Youtube.
- $n^*(w, 0, 0)$ is independent of w : a flat labor supply curve;
 - under log utility, income effect (+) and substitution effect (−) cancels out; see PS1.

How Taxes Distort Consumer Behavior?

Suppose the Congress prohibits the government from charging two types of taxes simultaneously. The government has to give up one tax instrument. Two questions:

- 1 Can you characterize consumer's behavior when government can charge either proportional tax or lump-sum tax?
- 2 Suppose the government wants to claim same amount of tax revenue from the consumer. Which tax regime makes the consumer better off?

Consumer Behavior with Proportional Tax Only

- With proportional tax only,

$$\tau > 0 \text{ and } T = 0$$

so that

$$\begin{aligned}c^P \equiv c^*(w, \tau, 0) &= \frac{wh(1 - \tau)}{1 + \gamma} \\l^P \equiv l^*(w, \tau, 0) &= \frac{\gamma}{1 + \gamma}h \\n^P \equiv n^*(w, \tau, 0) &= \frac{1}{1 + \gamma}h\end{aligned}$$

- Under log utility, the labor/leisure choice is not distorted by the proportional tax (again, IE and SE cancels out).
- Consumption is scaled down by a proportion of $(1 - \tau)$.

Consumer Behavior with Lump-sum Tax Only

- With lump-sum tax only,

$$\tau = 0 \text{ and } T > 0$$

so that

$$c^{LS} \equiv c^*(w, 0, T) = \frac{wh - T}{1 + \gamma}$$

$$l^{LS} \equiv l^*(w, 0, T) = \frac{\gamma}{1 + \gamma}h - \frac{\gamma T}{(1 + \gamma)w}$$

$$n^{LS} \equiv n^*(w, 0, T) = \frac{1}{1 + \gamma}h + \frac{\gamma T}{(1 + \gamma)w}$$

- Compared to the no-tax benchmark,
 - Consumption and leisure goes down (here, solely income effect);
 - Labor supply shifts up.

Graphical Illustration

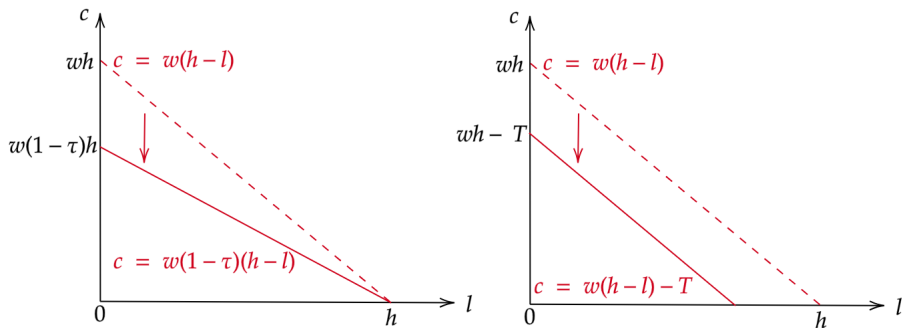


Figure: Proportional Tax (Left) vs Lump-sum Tax (Right)

Graphical Illustration

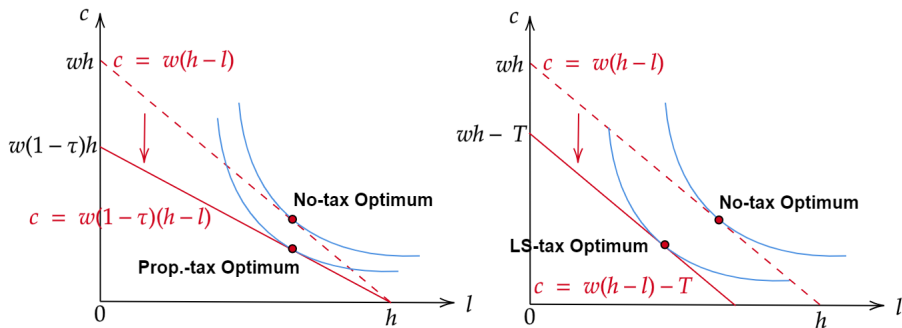


Figure: Proportional Tax (Left) vs Lump-sum Tax (Right)

Tax instruments shape consumer behavior by changing their budget constraint:

- Lump-sum tax is a pure income effect;
slope of budget constraint is not changed; parallel shift from no-tax benchmark.
- Proportional tax is akin to a wage-change exercise:
slope of budget constraint becomes flatter;
both substitution and income effect matter.

Welfare Analysis: Lump-sum vs Proportional

- Social welfare is measured by the maximum consumer utility:

$$W^{LS} \equiv u(c^{LS}, l^{LS})$$

$$W^P \equiv u(c^P, l^P)$$

- The government wants to claim same amount of tax revenue from the consumer, i.e.

$$T = \tau w(h - l^P)$$

Welfare Analysis: Graphical

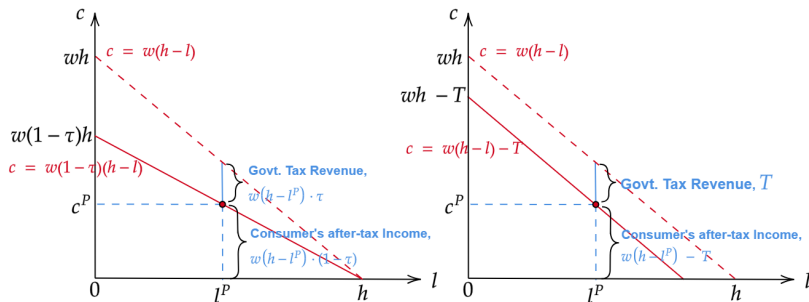


Figure: Govt. Tax Revenue: Prop. (left) vs LS (right)

Welfare Analysis: Graphical

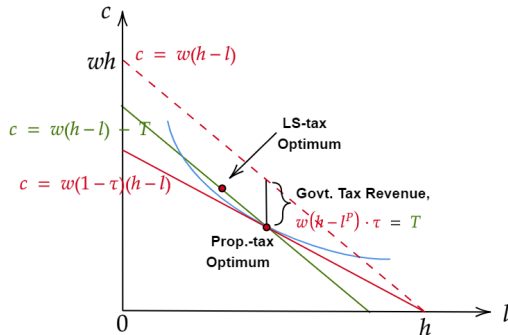


Figure: Lump-sum-tax Optimum Dominates Proportional-tax Optimum

Welfare Analysis: Graphic

- We've shown graphically that consumers are better off under lump-sum tax than proportional tax.
- can think of proportional tax optimum $\rightarrow$ lump-sum tax optimum as a wage-increase exercise.
- wage increase from $w(1 - \tau)$ to w :
expanding the budget set on the left of prop.-tax optimum
 $\implies$ higher utility for consumer.

Welfare Analysis: Algebraic

- Social welfare is measured by the maximum consumer utility:

$$W^{LS} \equiv u(c^{LS}, l^{LS}) = \log c^{LS} + \gamma \log l^{LS}$$

$$W^P \equiv u(c^P, l^P) = \log c^P + \gamma \log l^P$$

- The government wants to claim same amount of tax revenue from the consumer, i.e.

$$T = \tau w(h - l^P) = \tau w \cdot \frac{h}{1 + \gamma}$$

Plug it into the lump-sum tax decision rules:

$$c^{LS} = \frac{wh - \tau w \cdot \frac{h}{1 + \gamma}}{1 + \gamma} = \frac{wh}{1 + \gamma} \left(1 - \frac{\tau}{1 + \gamma}\right)$$

$$l^{LS} = \frac{\gamma}{1 + \gamma} h - \frac{\gamma \tau w \cdot \frac{h}{1 + \gamma}}{(1 + \gamma)w} = \frac{\gamma}{1 + \gamma} h \left(1 - \frac{\tau}{1 + \gamma}\right)$$

Welfare Analysis: Algebraic

- Now we have

$$c^{LS} = \frac{wh}{1+\gamma} \left(1 - \frac{\tau}{1+\gamma}\right)$$
$$I^{LS} = \frac{\gamma}{1+\gamma} h \left(1 - \frac{\tau}{1+\gamma}\right)$$

and

$$c^P = \frac{wh(1-\tau)}{1+\gamma}$$
$$I^P = \frac{\gamma}{1+\gamma} h$$

We simply plug them into the welfare functions:

Welfare Analysis: Algebraic

$$\begin{aligned} W^{LS} - W^P &= \log \left(\frac{wh}{1+\gamma} \left(1 - \frac{\tau}{1+\gamma} \right) \right) + \gamma \log \left(\frac{\gamma}{1+\gamma} h \left(1 - \frac{\tau}{1+\gamma} \right) \right) \\ &\quad - \log \left(\frac{wh(1-\tau)}{1+\gamma} \right) - \gamma \log \left(\frac{\gamma}{1+\gamma} h \right) \\ &= \log \left(1 - \frac{\tau}{1+\gamma} \right) + \gamma \log \left(1 - \frac{\tau}{1+\gamma} \right) - \log(1-\tau) \\ &= (1+\gamma) \log \left(1 - \frac{\tau}{1+\gamma} \right) - \log(1-\tau) \\ &> (1+0) \log \left(1 - \frac{\tau}{1+0} \right) - \log(1-\tau) = 0 \end{aligned}$$

Lump-sum tax is better for the consumer.